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THE ANALYSIS OF THE SEMI-INFINITE CYLINDER PROBLEM

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Abstract—A semi-infinite cylinder with fixed short end is considered. Normal loads far away from the fixed end are prescribed. An exact formulation of the problem in terms of a singular integral equation is provided by using an integral transform technique. Stresses along the rigid end and stress intensity factors are computed numerically and are presented graphically.

INTRODUCTION

Numerous studies have been devoted to the analysis of semi-infinite cylinders with stress-free curved surfaces and prescribed stress or displacement boundary conditions on the plane end. A few good solutions exist; however, none of the methods provides a solution which can directly give the correct behavior of stresses near the circumference of the short end without presenting convergence difficulties. The best solution known so far is given by Benthem and Minderhoud[1] in which the eigenfunction technique is utilized with remarkable success. No difficulty in the convergence of results is experienced; however, a prior knowledge of the proper stress singularities from alternate means is necessary. Also, an extensive review of earlier works attempting to solve the problem is provided in[1] and, hence, will not be repeated here.

In this paper, an integral transform technique is used to formulate the problem in terms of a singular integral equation which is then solved numerically. (This technique has been recently used by the author to solve the semi-infinite strip problem[2].) First an elastostatic problem for a penny-shaped inclusion located centrally in an infinite cylinder with stress-free curved boundaries is considered (Fig. 1). Inclusion extending to the surfaces then reduces the problem to that of a semi-infinite cylinder with fixed end.

FORMULATION OF PROBLEM

Consider the axisymmetric problem for a cylinder of radius R containing a rigid penny-shaped inclusion of radius a (Fig. 1a). Let E and ν be the Young's modulus and Poisson's ratio of the cylinder, respectively. The only loads acting on the cylinder are uniform stresses on the plane ends far away from the inclusion. The problem of interest here is the special case when the inclusion extends to the cylinder surface, i.e. $a = R$ (Fig. 1b). The formulation provided in this paper is valid for $a \leq R$.

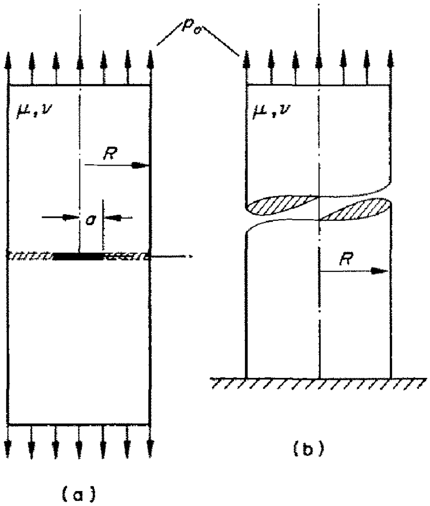

Fig. 1. Geometry of an infinite cylinder with a penny-shaped inclusion and a semi-infinite cylinder ($a = R$).

The problem described above can be recovered by superposing two subproblems as shown in Fig. 2. Solution of I is simply given as

$$\begin{aligned} \sigma_{rz}{}^I(r, z) &= \sigma_{rr}{}^I(r, z) = 0 \\ \sigma_{zz}{}^I(r, z) &= \sigma_{zz}{}^I(r, \pm\infty) = p_0 \\ u_r{}^I(r, 0) &= -\varepsilon_0 r, \qquad \varepsilon_0 = \frac{\nu}{E} p_0 \\ u_z{}^I(r, 0) &= 0. \end{aligned} \tag{1}$$

Problem II is the disturbance problem which, when added to I, must describe the original problem. Hence the input function for this part must be the radial displacement at the plane

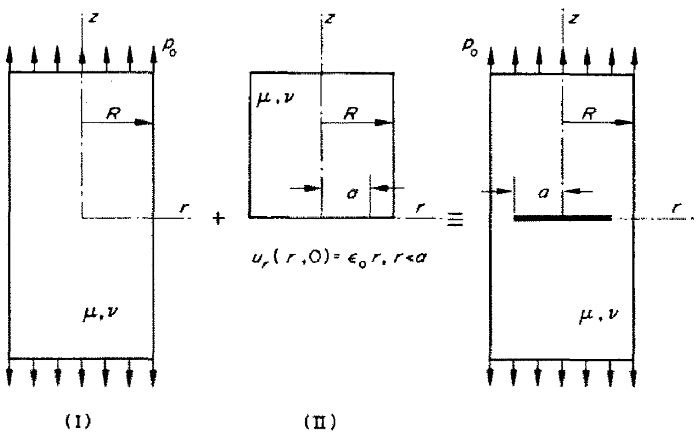

Fig. 2. Superposition of two solutions to give the total solution.

$z = 0$ and equal to the negative of that in I. Thus the boundary conditions for subproblem II become

$$\sigma_{rr}(R, z) = \sigma_{rz}(R, z) = 0$$
$$u_z(r, 0) = 0, \qquad r < R \tag{2}$$
$$u_r(r, 0) = \varepsilon_0 r, \qquad r < a$$
$$\sigma_{rz}(r, 0) = 0, \qquad a < r < R. \tag{3}$$

Note that $z = 0$, $r < a$ is a singular surface across which the displacements are continuous but the stress vector suffers a discontinuity. The displacements and stresses for the problem can be written as a superposition of two transform solutions. One is the solution for a cylinder ($r < R$, $|z| < \infty$) with $z = 0$ as the plane of symmetry and $r = 0$ as the axis of symmetry[3], and the other is the half plane solution in polar coordinates with $z = 0$ as the plane of symmetry[4]. The solution can be expressed as

$$\pi\mu u_r(r, z) = -\int_0^\infty [A_1 I_1(rt) + A_2 r I_0(rt)]t^2 \cos(zt)\,dt - \int_0^\infty B(\lambda)(\kappa - \lambda z)e^{-\lambda z}J_1(\lambda r)\,d\lambda$$

$$\pi\mu u_z(r, z) = \int_0^\infty [A_1 t I_0(rt) + A_2\{(\kappa + 1)I_0(rt) + rtI_1(rt)\}]t \sin(zt)\,dt + \int_0^\infty \lambda z B(\lambda)e^{-\lambda z}J_0(\lambda r)\,d\lambda$$

$$\begin{aligned}
\sigma_{rr}(r, z) = &-\frac{2}{\pi}\int_0^\infty \left[A_1 t\left\{I_0(rt) - \frac{I_1(rt)}{rt}\right\} + A_2\left\{\frac{\kappa - 1}{2} I_0(rt) + rtI_1(rt)\right\}\right]t^2 \cos(zt)\,dt \\
&+ \frac{2}{\pi}\int_0^\infty \left[\frac{\kappa J_1(\lambda r)}{\lambda r} - \frac{\kappa + 3}{2} J_0(\lambda r) + \lambda z\left\{J_0(\lambda r) - \frac{J_1(\lambda r)}{\lambda r}\right\}\right]\lambda B(\lambda)e^{-\lambda z}\,d\lambda \\
\sigma_{zz}(r, z) = &\frac{2}{\pi}\int_0^\infty \left[A_1 t I_0(rt) + A_2\left\{\frac{5 + \kappa}{2} I_0(rt) + rtI_1(rt)\right\}\right]t^2 \cos(zt)\,dt \\
&+ \frac{2}{\pi}\int_0^\infty \left[\frac{\kappa - 1}{2} - \lambda z\right]\lambda B(\lambda)e^{-\lambda z}J_0(\lambda r)\,d\lambda \\
\sigma_{rz}(r, z) = &\frac{2}{\pi}\int_0^\infty \left[A_1 t I_1(rt) + A_2\left\{\frac{1 + \kappa}{2} I_1(rt) + rtI_0(rt)\right\}\right]t^2 \sin(zt)\,dt \\
&+ \frac{2}{\pi}\int_0^\infty \left[\frac{\kappa + 1}{2} - \lambda z\right]\lambda B(\lambda)e^{-\lambda z}J_1(\lambda r)\,d\lambda
\end{aligned} \tag{4}$$

where $\kappa = 3 - 4\nu$ and $\mu = \dfrac{E}{2(1 + \nu)}$.

It should be noted that this solution identically satisfies the third condition $u_z(r, 0) = 0$ of equation (2). The three unknown functions $A_1(t)$, $A_2(t)$ and $B(\lambda)$ will be determined by using the first two conditions of equation (2) and the mixed boundary conditions, equation (3). The first two conditions of equation (2) can be written as

$$\begin{aligned}
&A_1 t^3\left\{I_0(Rt) - \frac{I_1(Rt)}{Rt}\right\} + A_2 t^2\left\{\frac{\kappa - 1}{2} I_0(Rt) + RtI_1(Rt)\right\} \\
&\quad = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2 B(\lambda)}{\lambda^2 + t^2}\left[\kappa\frac{J_1(\lambda R)}{\lambda R} - \frac{\kappa + 3}{2} J_0(\lambda R) + \frac{\lambda^2 - t^2}{\lambda^2 + t^2}\left\{J_0(\lambda R) - \frac{J_1(\lambda R)}{\lambda R}\right\}\right]d\lambda
\end{aligned} \tag{5}$$

$$A_1 t^3 I_1(Rt) + A_2 t^2\left\{\frac{1 + \kappa}{2} I_1(Rt) + RtI_0(Rt)\right\} = -\frac{2}{\pi}\int_0^\infty \lambda B(\lambda)\frac{J_1(\lambda R)}{\lambda^2 + t^2}\left[\frac{\kappa + 1}{2} - \frac{2\lambda^2}{\lambda^2 + t^2}\right]t\,d\lambda.$$

Mixed boundary conditions equation (3) can be expressed as

$$\pi\mu u_r(r, 0) = -\int_0^\infty \kappa B(\lambda)J_1(\lambda r)\,d\lambda - \int_0^\infty [A_1 I_1(rt) + A_2 r I_0(rt)]t^2\,dt = \pi\mu\varepsilon_0 r, \qquad r < a \quad (6)$$

$$\sigma_{rz}(r, 0) = \frac{2}{\pi}\int_0^\infty \frac{\kappa+1}{2}\lambda B(\lambda)J_1(\lambda r)\,d\lambda = 0, \qquad a < r < R. \quad (7)$$

The dual integral equations written above can be reduced to a singular integral equation by defining a new unknown function $\phi(r)$ representing the shear stress along the line $z = 0^+$. Note that this must be distinguished from the shear stress at $z = 0^-$ since the stress vector suffers a discontinuity across this plane. Due to symmetry, the two values will be numerically equal and the negative of each other. The function $\phi(r)$ is written as

$$\phi(r) = \sigma_{rz}(r, 0^+), \qquad r < R. \quad (8)$$

From equation (7) $\phi(r) = 0$, $a < r < R$. Hence inverting the integral obtained from equation (7) and equation (8) we have

$$\frac{\kappa+1}{\pi}B(\lambda) = \int_0^\infty \phi(\rho)\rho J_1(\lambda\rho)\,d\rho. \quad (9)$$

Equations (5) are now solved simultaneously to give

$$A_1 t^2 = \frac{RD_1(t)\left[\frac{\kappa+1}{2}I_1(Rt) + RtI_0(Rt)\right] - RD_2(t)\left[\frac{\kappa-1}{2}I_0(Rt) + RtI_1(Rt)\right]}{R^2t^2I_0^2(Rt) - \left[\frac{\kappa+1}{2} + R^2t^2\right]I_1^2(Rt)}$$

$$A_2 t^2 = \frac{D_2(t)[RtI_0(Rt) - I_1(Rt)] - RtD_1(t)I_1(Rt)}{R^2t^2I_0^2(Rt) - \left[\frac{\kappa+1}{2} + R^2t^2\right]I_1^2(Rt)} \quad (10)$$

where

$$D_1(t) = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2 B(\lambda)}{\lambda^2+t^2}\left[\left\{(\kappa-1) + \frac{2t^2}{\lambda^2+t^2}\right\}\frac{J_1(\lambda R)}{\lambda R} - \left\{\frac{\kappa+1}{2} + \frac{2t^2}{\lambda^2+t^2}\right\}J_0(\lambda R)\right]d\lambda$$

$$D_2(t) = -\frac{2}{\pi}\int_0^\infty \lambda t\frac{B(\lambda)}{\lambda^2+t^2}\left[\frac{\kappa-3}{2} + \frac{2t^2}{\lambda^2+t^2}\right]J_1(\lambda R)\,d\lambda. \quad (11)$$

Using equation (9) and relations given in (5), equations (11) can be reduced to give the following expressions.

$$\frac{\kappa+1}{2}D_i(t) = \int_0^a \phi(\rho)k_i(\rho, t)\rho\,d\rho, \qquad i = 1, 2, \ldots. \quad (12)$$

where

$$k_1(\rho, t) = \frac{1}{R}\Big[(\kappa + 1 + R^2t^2)I_1(\rho t)K_1(Rt) + \frac{\kappa+3}{2}RtI_1(\rho t)K_0(Rt)$$
$$- \rho tI_0(\rho t)K_1(Rt) - \rho tRtI_0(\rho t)K_0(Rt)\Big] \quad (13)$$

$$k_2(\rho, t) = -t\left[RtI_1(\rho t)K_0(Rt) - \rho tI_0(\rho t)K_1(Rt) + \frac{\kappa+1}{2}I_1(\rho t)K_1(Rt)\right].$$

Now A_1 and A_2 are substituted from equation (10) into equation (6). Equation (9) is used to eliminate $B(\lambda)$, and equation (6) becomes an integral equation containing the only unknown function $\phi(r)$. In order to reduce this to a singular integral equation, instead of radial displacement u_r, equation (6) is rewritten for $(1/r)(\partial/\partial r)(ru_r)$ which is equal to the sum of radial and tangential strains at the plane $z = 0$[6]. The inner integral appearing due to the first term in equation (6) can be written in closed form in terms of complete elliptic integrals. The singular integral equation can be expressed as

$$\int_0^a \phi(\rho)(\kappa h(\rho, r) + \rho L(\rho, r)]\,\mathrm{d}\rho = -\mu(1+\kappa)\frac{\pi}{2}\varepsilon_0, \qquad r < a \tag{14}$$

where

$$h(\rho, r) = \begin{cases} \dfrac{\rho}{\rho^2 - r^2} E\left(\dfrac{r}{\rho}\right), & \rho < r \\[2ex] \dfrac{r}{\rho^2 - r^2} E\left(\dfrac{\rho}{r}\right) + \dfrac{1}{r} K\left(\dfrac{\rho}{r}\right), & \rho > r \end{cases}$$

$$L(\rho, r) = \int_0^\infty \frac{k_1(\rho, t)h_1(r, t) + k_2(\rho, t)h_2(r, t)}{R^2t^2 I_0^{\,2}(Rt) - \left[\dfrac{\kappa+1}{2} + R^2t^2\right] I_1^{\,2}(Rt)}\,\mathrm{d}t \tag{15}$$

$$h_1(r, t) = Rt\left[RtI_0(rt)I_0(Rt) - rtI_1(rt)I_1(Rt) + \frac{\kappa - 3}{2} I_0(rt)I_1(Rt)\right]$$

$$-h_2(r, t) = Rt\left[RtI_0(rt)I_1(Rt) - rtI_1(rt)I_0(Rt) + \frac{\kappa - 5}{2} I_0(rt)I_0(Rt)\right] + rtI_1(rt)I_1(Rt) + 2I_0(rt)I_1(Rt)$$

E and K being the complete elliptic integrals of the first and second kind respectively. Note that the kernel $h(\rho, r)$ behaves like a Cauchy kernel $1/\rho - r$ near $\rho \to r$. The kernel $L(\rho,r)$ is bounded for all values of ρ and r in $(0, a)$ if $a < R$. For the case of $a < R$, the solution of equation (14) is quite straightforward[4]. It is convenient for the purpose of numerical analysis to extend the definition of the kernels into the negative range $(-a, 0)$ by assuming that $\phi(\rho) = -\phi(-\rho)$ to obtain

$$\int_{-a}^a \phi(\rho)\left[\frac{\kappa}{\rho - r} + \kappa b(\rho, r) + |\rho| L(\rho, r)\right]\mathrm{d}\rho = -\mu(1+\kappa)\pi\varepsilon_0, \qquad |r| < a \tag{16}$$

where

$$b(\rho, r) = \frac{m_1(\rho, r) - 1}{\rho - r}$$

$$m_1(\rho, r) = \begin{cases} E\left(\dfrac{r}{\rho}\right), & |\rho| < |r| \\[2ex] \left|\dfrac{r}{\rho}\right| E\left(\dfrac{\rho}{r}\right) + \dfrac{\rho^2 - r^2}{|\rho r|} K\left(\dfrac{\rho}{r}\right), & |\rho| > |r|. \end{cases} \tag{17}$$

Note that $b(\rho, r)$ is a bounded kernel in the domain $-a < \rho, r < a$ for $a \leq R$. Equation (16) should be solved subject to the following consistency condition

$$\int_{-a}^{a} \phi(\rho)\,\mathrm{d}\rho = 0. \tag{18}$$

The actual problem of interest here (Fig. 1b) is the case when $a = R$ and the kernel $L(\rho, r)$ is no longer bounded for all values of ρ and r, and contains point singularities at $\rho = \pm R$ and $r = \pm R$. These singularities can be extracted by using the asymptotic value of the integrand appearing in the infinite integral for the kernel $L(\rho, r)$, as $t \to \infty$. Let

$$K_s(\rho, r) = \int_0^{\infty} k_{\infty}(\rho, r, t)\,\mathrm{d}t \tag{19}$$

where $K_s(\rho, r)$ is the singular part of the kernel $L(\rho, r)$. Using the asymptotic behavior of the modified Bessel functions[7], from the second equation in equation (15), for positive values of ρ and r, it follows that

$$k_{\infty}(\rho, r, t) = \frac{1}{\sqrt{(\rho r)}} e^{-(2R-\rho-r)t}\Big[(R-\rho)(R-r)t^2 + \{\kappa(R-r) + (\kappa-2)(R-\rho)\}\frac{t}{2} + \frac{(\kappa-1)^2}{4}\Big]. \tag{20}$$

Using equation (20) and performing the integration in equation (19), the corresponding singular part of the kernel for $(\rho, r) > 0$ becomes

$$K_{1s}(\rho, r) = \frac{1}{4\sqrt{(\rho r)}}\left[\frac{\kappa^2 - 3}{2R-\rho-r} + \frac{12(R-r)}{(2R-\rho-r)^2} - \frac{8(R-r)^2}{(2R-\rho-r)^3}\right]. \tag{21}$$

Then

$$K_s(\rho, r) = K_{1s}(\rho, r) - K_{1s}(-\rho, r) + K_{1s}(\rho, -r) - K_{1s}(-\rho, -r). \tag{22}$$

Note that the additional singular terms appear due to the extension of equation (14) into the negative range $(-R, 0)$.

In order to analyze the behavior of the unknown function $\phi(\rho)$ near the end points $\rho = \pm R$, the dominant part of equation (16) (for $a = R$) consisting of the Cauchy kernel and the singular kernel $K_s(\rho, r)$ must be considered. This can be expressed as

$$\begin{aligned}
&\frac{1}{\pi}\int_{-R}^{R}\phi(\rho)\Big[\frac{\kappa}{\rho - r} + \frac{1}{4}\sqrt{\left(\frac{\rho}{r}\right)}\Big\{\kappa^2 - 3 + 12(R-r)\frac{\mathrm{d}}{\mathrm{d}r} - 4(R-r)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big\}\\
&\quad\times\Big\{\frac{1}{2R-\rho-r} - \frac{1}{2R+\rho-r}\Big\} + \frac{1}{4}\sqrt{\left(\frac{\rho}{r}\right)}\Big\{\kappa^2 - 3 - 12(R+r)\frac{\mathrm{d}}{\mathrm{d}r} - 4(R+r)^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big\}\\
&\quad\times\Big\{\frac{1}{2R-\rho+r} - \frac{1}{2R+\rho+r}\Big\}\Big]\,\mathrm{d}\rho\\
&= -\mu(1+\kappa)\varepsilon_0 + A(r), \qquad |r| < R
\end{aligned} \tag{23}$$

where $A(r)$ is the bounded function containing the terms coming from the Fredholm kernel in equation (17), i.e.

$$A(r) = -\int_{-R}^{R} \phi(\rho)[\kappa b(\rho, r) + |\rho|\{L(\rho, r) - K_s(\rho, r)\}]\, \mathrm{d}\rho \tag{24}$$

$\phi(\rho)$ may be assumed to have integrable singularity at $\rho = \pm R$ which can be expressed as[8]

$$\phi(\rho) = \frac{G(\rho)}{(R^2 - \rho^2)^\alpha} = \frac{G(\rho)e^{(\pi i\alpha)}}{(\rho - R)^\alpha(\rho + R)^\alpha}, \qquad |\rho| < R \tag{25}$$

where $0 < Re(\alpha) < 1$ and $G(\rho)$ satisfies a Hölder condition in the closed interval $|\rho| \leq R$. The method of determining α requires studying equation (23) as in ([8], chap. 4) and has been discussed in detail in[9].

Considering the sectionally holomorphic function

$$\psi(z) = \frac{1}{\pi}\int_{-R}^{R} \frac{\phi(\rho)}{\rho - z}\, \mathrm{d}\rho = \frac{1}{\pi}\int_{-R}^{R} \frac{G(\rho)\mathrm{e}^{(\pi i\alpha)}}{(\rho - R)^\alpha(\rho + R)^\alpha} \frac{\mathrm{d}\rho}{\rho - z} \tag{26}$$

and following [8], one obtains

$$\begin{aligned}
\psi(z) &= \frac{G(-R)}{(2R)^\alpha} \frac{e^{(\pi i\alpha)}}{\sin \pi\alpha} \frac{1}{(z + R)^\alpha} - \frac{G(R)}{(2R)^\alpha} \frac{1}{\sin \pi\alpha(z - R)^\alpha} + \phi_0(z) \\
\psi(r) &= \frac{G(-R)}{(2R)^\alpha} \frac{\cot \pi\alpha}{(R + r)^\alpha} - \frac{G(R)}{(2R)^\alpha} \frac{\cot \pi\alpha}{(R - r)^\alpha} + \phi_1(r), \qquad |r| < R \\
\psi(2R - r) &= -\frac{G(R)}{(2R)^\alpha} \frac{1}{\sin \pi\alpha(R - r)^\alpha} + \phi_2(r), \qquad R < 2R - r < 3R \\
\psi(-2R + r) &= \frac{G(-R)}{(2R)^\alpha} \frac{1}{\sin \pi\alpha(R - r)^\alpha} + \phi_3(r), \qquad -3R < r - 2R < -R \\
\psi(2R + r) &= -\frac{G(R)}{(2R)^\alpha} \frac{1}{\sin \pi\alpha(R + r)^\alpha} + \phi_4(r), \qquad R < 2R + r < 3R \\
\psi(-2R - r) &= \frac{G(-R)}{(2R)^\alpha} \frac{1}{\sin \pi\alpha(R + r)^\alpha} + \phi_5(r), \qquad -3R < -r - 2R < -R
\end{aligned} \tag{27}$$

where $\phi_i(r)$, $i = 0, \ldots, 5$ are bounded everywhere except possibly at the end points $\pm R$ where it may have the following behavior:

$$|\phi_i(r)| < \frac{G_0(\pm R)}{(R \pm r)^{\alpha_0}}, \qquad Re(\alpha_0) < Re(\alpha). \tag{28}$$

Substituting equation (27) into equation (23) and using the symmetric properties of the function $G(\rho)$, equation (23) becomes

$$\frac{1}{(2R)^\alpha \sin \pi\alpha} [\kappa \cos \pi\alpha - \tfrac{1}{2}\{(\kappa^2 - 3) + 12\alpha - 4\alpha(\alpha + 1)\}] \cdot \left[\frac{G(-R)}{(R + r)^\alpha} - \frac{G(R)}{(R - r)^\alpha}\right] = P(r) \tag{29}$$

where $P(r)$ contains all the bounded functions. Since $G(\pm R) \neq 0$, equation (29) can only be satisfied if

$$2\kappa \cos \pi\alpha = \kappa^2 + 1 - 4(\alpha - 1)^2 \tag{30}$$

which provides a characteristic equation to determine α[2]. Note that the equation is identical to that of a 90° wedge with stress-displacement boundary conditions. Also, it depends only on the Poisson's ratio of the cylinder and is real for any material, i.e. for $0 \leq \nu \leq 0{\cdot}5$. If $a < R$ (Fig. 1a), equation (23) would only have a Cauchy kernel and the characteristic equation would become

$$\cot \pi\alpha = 0, \qquad \alpha = \tfrac{1}{2} \tag{31}$$

which is the well known singularity along the circumference of a penny-shaped rigid inclusion embedded in a material. For the semi-infinite cylinder, the singular integral equation can now be written as

$$\int_{-R}^{R} \phi(\rho)\left[\frac{\kappa}{\rho - r} + \kappa b(\rho, r) + |\rho| K_s(\rho, r) + |\rho| K_F(\rho, r)\right] \mathrm{d}\rho = -\mu(\kappa + 1)\varepsilon_0\pi \qquad |r| < R$$
$$K_F(\rho, r) = L(\rho, r) - \int_0^{\infty} [k_\infty(\rho, r, t) - k_\infty(-\rho, r, t) + k_\infty(\rho, -r, t) - k_\infty(-\rho, -r, t]\,\mathrm{d}t \tag{32}$$

where $b(\rho, r)$, $K_s(\rho, r)$, $L(\rho, r)$ and $k_\infty(\rho, r, t)$ are given by equation (17), equation (22), equation (15) and equation (20) respectively. $K_F(\rho, r)$ is the Fredholm kernel for $a \leq R$.

SOLUTION OF THE INTEGRAL EQUATION

Equation (32) is first normalized with respect to R by using the following transformation:

$$\tau = \frac{\rho}{R}, \qquad y = \frac{r}{R}, \qquad \phi(\rho) = \phi(\tau R) = \phi(\tau). \tag{33}$$

Hence equation (32) can now be written as

$$\int_{-1}^{1} \phi(\tau)\left[\frac{\kappa}{\tau - y} + \kappa R b(R\tau, Ry) + R^2|\tau|\{K_s(R\tau, Ry) + K_F(R\tau, Ry)\}\right] \mathrm{d}\tau = -\mu(\kappa + 1)\varepsilon_0\pi, \qquad |y| < 1 \tag{34}$$

and equation (25) becomes

$$\phi(\tau) = \frac{\psi(\tau)}{(1 - \tau^2)^\alpha} \tag{35}$$

where α is given by equation (30). Equation (34) can now be solved numerically by using Gauss–Jacobi Integration formula. The technique has been used and discussed in detail in [9]. Using the additional condition equation (18), the following set of $N \times N$ simultaneous algebraic equations are obtained:

$$\sum_{j=1}^{N} A_j \psi(\tau_j)\left[\frac{\kappa}{\tau_j - y_i} + \kappa R b(R\tau_j, Ry_i) + R^2|\tau_j|\{K_s(R\tau_j, Ry_i) + K_F(R\tau_j, Ry_i)\}\right] = -\mu(\kappa + 1)\pi\varepsilon_0, \qquad i = 1, \ldots, N - 1 \tag{36}$$
$$\sum_{j=1}^{N} A_j \psi(\tau_j) = 0$$

where

$$P_N^{(-\alpha, -\alpha)}(\tau_j) = 0, \qquad j = 1, \ldots, N$$
$$P_{N-1}^{(1-\alpha, 1-\alpha)}(y_i) = 0, \qquad i = 1, \ldots, N - 1$$

and A_j are the corresponding weighting constants[9]. $\psi(\tau_j)$ are computed numerically from equation (36), and the shear stress $\sigma_{rz}(r, 0)$ can then be expressed as

$$\sigma_{rz}(r, 0) = \phi(r) = \frac{R^{2\alpha}\psi\left(\frac{h}{r}\right)}{(R^2 - r^2)^\alpha}, \qquad |r| < R. \tag{37}$$

NORMAL STRESS AND STRESS INTENSITY FACTOR

Having solved for the shear stress $\sigma_{rz}(r, 0)$, the stress and the displacement field at any location in the semi-infinite cylinder can be calculated by using the appropriate equation in equation (4). The normal stress $\sigma_{zz}(r, 0)$ is of particular importance. To evaluate σ_{zz}, the fourth equation in equation (4) must be considered. Using equations (9), (10) and (12), it can be expressed as

$$\pi(\kappa + 1)\sigma_{zz}(r, 0) = \int_{-R}^{R} \phi(\rho)\left[\frac{\kappa - 1}{\rho - r} + (\kappa - 1)b(\rho, r) + |\rho|\{K_{2s}(\rho, r) + K_{1F}(\rho, r)\}\right] \mathrm{d}\rho \tag{38}$$

where

$$\begin{aligned}
K_{1F}(\rho, r) &= L_1(\rho, r) - \int_0^\infty [k_{1\infty}(\rho, r, t) - k_{1\infty}(-\rho, r, t) \\
&\quad + k_{1\infty}(\rho, -r, t) - k_{1\infty}(-\rho, -r, t)]\,\mathrm{d}t \\
L_1(\rho, r) &= \int_0^\infty \frac{k_1(\rho, t)h_3(r, t) + k_2(\rho, t)h_4(r, t)}{R^2t^2I_0^2(Rt) - \left[\frac{\kappa + 1}{2} + R^2t^2\right]I_1^2(Rt)}\,\mathrm{d}t \\
h_3(r, t) &= RtI_0(Rt)I_0(rt) - rtI_1(Rt)I_1(rt) - 2I_1(Rt)I_0(rt) \\
h_4(r, t) &= Rt[RtI_1(Rt)I_0(rt) - rtI_0(Rt)I_1(rt) - 3I_0(Rt)I_0(rt)] \\
&\quad + \frac{5 + \kappa}{2} I_1(Rt)I_0(rt) + rtI_1(Rt)I_1(rt) \\
k_{1\infty}(\rho, r, t) &= \frac{1}{\sqrt{(\rho r)}} e^{-(2R-\rho-r)t}\Big[2(R - \rho)(R - r)t^2 \\
&\quad + \{\kappa(R - r) - 3(R - \rho)\}t - \frac{3\kappa - 1}{2}\Big] \\
K_{2s}(\rho, r) &= K_{3s}(\rho, r) - K_{3s}(-\rho, r) + K_{3s}(\rho, -r) - K_{3s}(-\rho, -r) \\
K_{3s}(\rho, r) &= \frac{1}{2\sqrt{(\rho r)}}\left[-\frac{3\kappa + 5}{2R - \rho - r} + \frac{2(\kappa + 7)(R - r)}{(2R - \rho - r)^2} - \frac{8(R - r)^2}{(2R - \rho - r)^3}\right].
\end{aligned} \tag{39}$$

In order to determine the behavior of the normal stress near the end points $\rho = \pm R$, again the dominant part of equation (38) consisting of the Cauchy kernel and the singular kernel $K_{2s}(\rho, r)$ must be analyzed in the same fashion as shown in the previous section. Using the relations of equation (27), the dominant part of the normal stress becomes

$$\begin{aligned}
(\kappa + 1)\sigma_{zz}(r, 0) &= \frac{1}{(2R)^\alpha \sin \pi\alpha} [(\kappa - 1)(\cos \pi\alpha + 1) - 2(\kappa + 1)(\alpha - 1) \\
&\quad + 4(\alpha - 1)^2]\left[\frac{G(-R)}{(R + r)^\alpha} - \frac{G(R)}{(R - r)^\alpha}\right], \qquad r \to \pm R
\end{aligned} \tag{40}$$

Stress intensity factors K_1 and K_2 are defined as

$$K_1 = \lim_{r \to R} \sqrt{2}(R - r)^{\alpha} \sigma_{zz}(r, 0)$$
$$K_2 = \lim_{r \to R} \sqrt{2}(R - r)^{\alpha} \sigma_{rz}(r, 0). \tag{41}$$

Using equation (37) and equation (40), these can be rewritten as

$$K_1 = -\frac{\sqrt{2}\, G(R)}{(\kappa + 1)(2R)^{\alpha} \sin \pi\alpha} [(\kappa - 1)(\cos \pi\alpha + 1) - 2(\kappa + 1)(\alpha - 1) + 4(\alpha - 1)^2]$$
$$K_2 = \frac{\sqrt{2}\, G(R)}{(2R)^{\alpha}}. \tag{42}$$

NUMERICAL RESULTS AND DISCUSSION

The total solution of the problem described in Fig. 1b is now the superposition of two problems *I* and *II*. Hence,

$$\sigma_{zz}^{T}(r, 0) = \sigma_{zz}^{I}(r, 0) + \sigma_{zz}(r, 0) = p_0 + \sigma_{zz}(r, 0)$$
$$\sigma_{rz}^{T}(r, 0) = \sigma_{rz}^{I}(r, 0) + \sigma_{rz}(r, 0) = \sigma_{rz}(r, 0). \tag{43}$$

Figures 3 and 4 show the variation of the shear and normal stresses, respectively, along the plane end for various values of the cylinder Poisson's ratio. Note that as the Poisson's ratio goes to zero, the disturbance problem *II* ceases to exist, and the total solution of the problem reduces to that of problem *I*, i.e. for $\nu = 0$

$$\sigma_{rx}^{T}(r, 0) = \sigma_{rz}^{I}(r, z) = 0$$
$$\sigma_{zz}^{T}(r, 0) = \sigma_{zz}^{I}(r, z) = p_0. \tag{44}$$

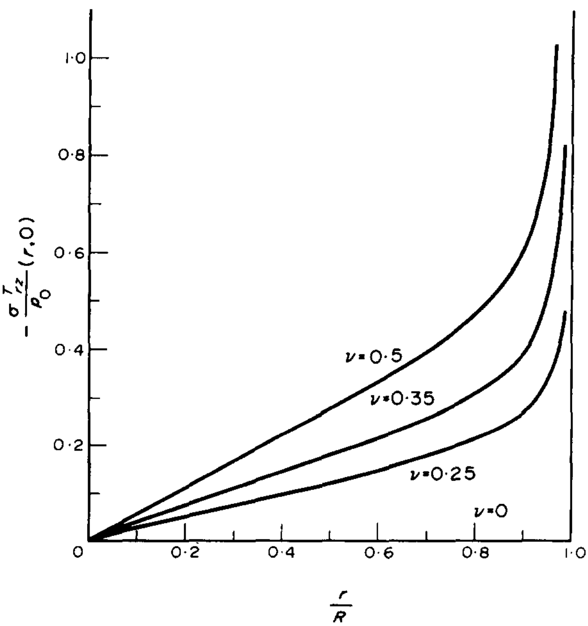

Fig. 3. Shear stress vs Poisson's ratio for the semi-infinite cylinder.

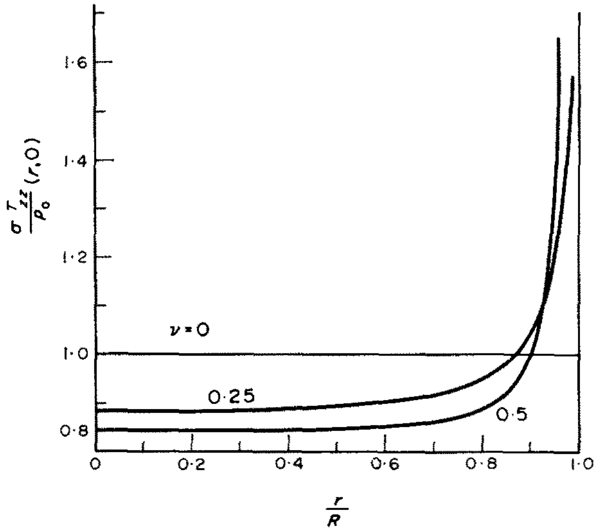

Fig. 4. Normal stress vs Poisson's ratio for the semi-infinite cylinder.

With increase in the cylinder Poisson's ratio (to a maximum value of 0·5), effect of the disturbance problem increases as is illustrated by Figs. 3 and 4. Also, for higher value of ν, a larger value of the power of the stress singularity α is obtained.

Figure 5 shows the variation of the stress intensity factor K_2 with respect to the Poisson's ratio of the cylinder. The value of K_2 for $\nu = 0{\cdot}25$ and the corresponding stress distributions compare very well with those obtained by Benthem and Minderhoud[1]. Equations (42)

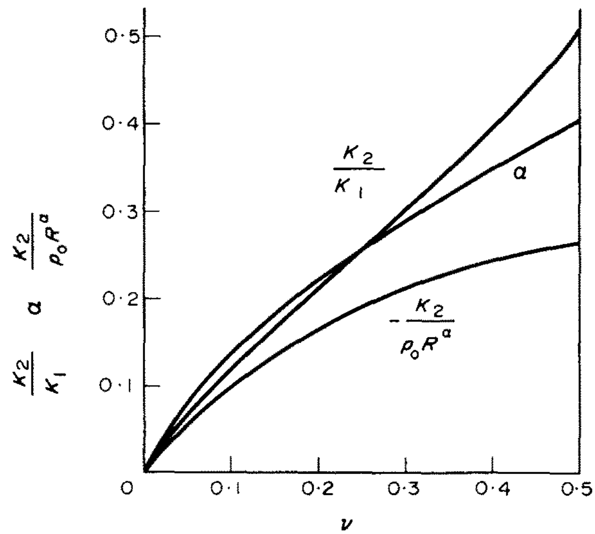

Fig. 5. Stress intensity factor $K_2/p_0 R^\alpha$, K_2/K_1 and the exponent α vs ν.

show that the stress intensity factors K_1 and K_2 are interdependent and their ratio K_2/K_1 depends only on the Poissons ratio of the cylinder. From equation (40)

$$\frac{K_2}{K_1} = \frac{(\kappa + 1)\sin \pi\alpha}{[(\kappa - 1)(\cos \pi\alpha + 1) - 2(\kappa + 1)(\alpha - 1) + 4(\alpha - 1)^2]}. \tag{45}$$

Figure 5 also shows a variation of this ratio K_2/K_1 with respect to ν. Note that α appearing in equation (45) is related to the cylinder Poisson's ratio ν as in equation (30). Also, the stress intensity factors as defined in equation (41) have a dimension different than usually encountered in crack problems. This is because $\alpha \neq 1/2$ and $K_i/(p_0 R^\alpha)$ is a dimensionless quantity. Figure 5 also shows a plot of this exponent α with respect to the cylinder Poisson's ratio ν.

The physical significance of the ratio K_2/K_1 can be appreciated by considering an elastic cylinder pressing against a much stiffer body (i.e. p_0 is negative). In this case if the coefficient of friction f between the two materials is greater than K_2/K_1, it may be assumed that there will be no sliding between the cylinder and the adjoining rigid body. So if $f > K_2/K_1$, the contact condition may be assumed to be that of perfect adhesion and the results given in this paper are valid. On the other hand, if $f < K_2/K_1$, the problem becomes that of an elastic cylindrical punch on a rigid half space with friction. Noting the variation K_2/K_1 vs. ν from Fig. 4 it is seen that, in testing whether the end condition is that of perfect adhesion or sliding for a given cylinder under compression, as a first approximation one may assume that $K_2/K_1 = \nu$. The exact expression, of course, is given by equation (45).

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Абстракт — Исследуется полубесконечный цилиндр, близкий конец которого защемлен. Нормальная нагрузка находится далеко от защемленного конца. Применяя метод интегрального преобразования, определяется строгая формулировка задачи, в виде сингулярного интегрального уравнения. Подсчитываются численно напряжения вдоль жесткого конца и факторы интенсивности напряжений. Представляются они в виде графиков.